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MATHEMATICS
TARGET IIT JEE

STRAIGHT LINE

THEORY AND EXERCISE BOOKLET

CONTENTS

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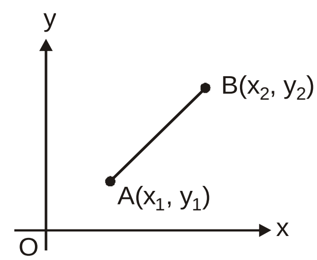
JEE Syllabus :

Cartesian coordinates, distance between two points, section formulae, shift of origin, equation of a straight line in various forms, angle between two lines, distance of a point from a line. Lines through the point of intersection of two given lines, equation of the bisector of the angle between two lines, concurrency of lines, centroid, orthocentre, incentre and circumcentre of a triangle.

A. COORDINATE GEOMETRY

Coordinate Geometry is the unification of algebra and geometry in which algebra is used in the study of geometrical relations and geometrical figures are represented by means of equations. The most popular coordinate system is the rectangular Cartesian system. Coordinates of a point are the real variables associated in an order to describe its location in space. Here we consider the space to be two-dimensional. Through a point O, referred to as the origin, we take two mutually perpendicular lines XOX' and YOY' and call them x and y axes respectively. The position of a point is completely determined with reference to these axes by means of an ordered pair of real numbers (x, y) called the coordinates of P where | x | and | y | are the distances of the point P from the y-axis and the x-axis respectively, x is called the x-coordinate or the abscissa of P and y is called the y-coordinate or the ordinate of the point P.

(1) Distance between two points :



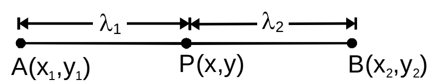
(a) Let A and B be two given points, whose coordinates are given by $A(x_1, y_1)$ and $B(x_2, y_2)$ respectively. Then AB = $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

(b) Distance of (x_1, y_1) from origin : $\sqrt{x_1^2 + y_1^2}$

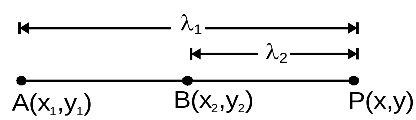
Note :- If two vertex $A(x_1, y_1)$, $B(x_2, y_2)$ are given then third vertex of equilateral triangle

$$C \text{ is } \left[\frac{x_1 + x_2 \mp \sqrt{3}(y_2 - y_1)}{2}, \frac{y_1 + y_2 \pm \sqrt{3}(x_2 - x_1)}{2}\right]$$

(2) Section formula :



Coordinates of the point P dividing the join of two points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the given ratio $\lambda_1 : \lambda_2$ are $P\left(\frac{\lambda_2 x_1 + \lambda_1 x_2}{\lambda_2 + \lambda_1}, \frac{\lambda_2 y_1 + \lambda_1 y_2}{\lambda_2 + \lambda_1}\right)$.



Coordinates of the point P dividing the join of two points $A(x_1, y_1)$ and $B(x_2, y_2)$ externally in the ratio of $\lambda_1 : \lambda_2$ are $P\left(\frac{\lambda_2 x_1 - \lambda_1 x_2}{\lambda_2 - \lambda_1}, \frac{\lambda_2 y_1 - \lambda_1 y_2}{\lambda_2 - \lambda_1}\right)$. In both the cases, λ_1/λ_2 is positive.

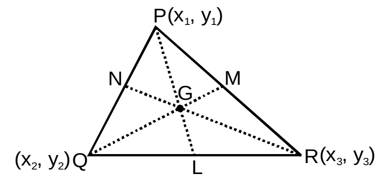
Notes :

(i) If the ratio, in which a given line segment is divided, is to be determined, then sometimes, for convenience (instead of taking the ratio $\lambda_1 : \lambda_2$), we take the ratio k : 1. If the value of k turns out to be positive, it is an internal division otherwise it is an external division.

(ii) The coordinates of the mid-point of the line-segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

(3) Special points in a triangle with co-ordinates :

(a) Centroid (G) :



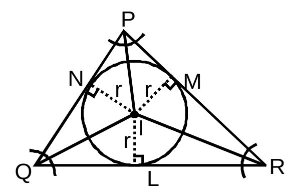
Definition : The point of concurrence of the medians of a triangle is called the centroid of the triangle.

(i) **G** divides median into 2 : 1.

(ii) **G** always lies inside the triangle.

(iii) Co-ordinates of **G** is $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$ or $\left(\frac{\sum x_1}{3}, \frac{\sum y_1}{3}\right)$

(b) Incentre (I) :



Definition : The point of concurrency of the internal bisectors of the angles of a triangle is called the incentre of the triangle.

(i) **I** always lies inside the triangle.

(ii) Internal angle bisector divides the base in the ratio of adjacent sides.

(iii) Co-ordinates of **I** is $\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$

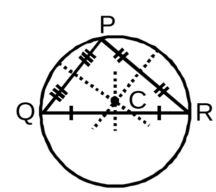
where a, b, c are the lengths of the sides of the Δ

(c) Ex-centres (I_1, I_2, I_3) :

Definition : The centre of the escribed circle which is opposite to vertices.

To get $\mathbf{I_1}$ (or $\mathbf{I_2}$ or $\mathbf{I_3}$) replace a by –a (b by –b or c by –c) in formula of coordinate of **I**

(d) Circumcentre (C) :



Definition : The point of concurrency of the perpendicular bisectors of the sides of a triangle is called circumcentre of the triangle.

(i) For acute angle $\Delta \Rightarrow$ lies inside

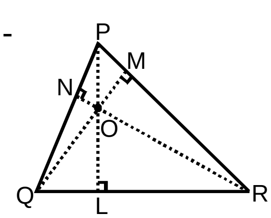
(ii) For obtuse angle $\Delta \Rightarrow$ lies outside

(iii) For right angle $\Delta \Rightarrow$ Mid point of hypotenuse

(iv) Co-ordinates of circumcentre is

$$\left(\frac{x_1\sin 2A+x_2\sin 2B+x_3\sin 2C}{\sin 2A+\sin 2B+\sin 2C}, \frac{y_1\sin 2A+y_2\sin 2B+y_3\sin 2C}{\sin 2A+\sin 2B+\sin 2C}\right)$$

(e) Orthocentre (O) :



Definition : The point of concurrency of the altitudes of a triangle is called the orthocentre of the triangle.

(i) For acute angle $\Delta \Rightarrow$ lies inside

(ii) For obtuse angle $\Delta \Rightarrow$ lies outside

(iii) For right angle $\Delta \Rightarrow$ vertex at $\perp^{ar}$

(iv) Co-ordinates of orthocentre is

$$\left(\frac{x_1\tan A+x_2\tan B+x_3\tan C}{\tan A+\tan B+\tan C}, \frac{y_1\tan A+y_2\tan B+y_3\tan C}{\tan A+\tan B+\tan C}\right)$$

Notes :

(i) In any triangle **O**, **G**, **C** are collinear.

(ii) In any triangle **G** divides the line joining **O** & **C** in ratio 2 : 1.

(iii) In an equilateral triangle **O**, **G**, **C**, **I** are coincident.

(iv) In an isosceles triangle **O**, **G**, **C**, **I** are collinear.

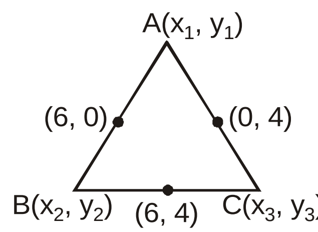
(f) Harmonic Conjugate : If P is a point that divides AB internally in the ratio $m_1 : m_2$ and Q is another point which divides AB externally in the same ratio $m_1 : m_2$, then the point P and Q are said to be Harmonic conjugate to each other with respect to A and B.

i.e. AP , AB and AQ forms a HP $\Rightarrow \frac{1}{AP} + \frac{1}{AQ} = \frac{2}{AB}$

Note :- Internal and External angle bisector of an angle divides the base harmonically.

Ex.1 If midpoints of the sides of a triangle are (0, 4), (6, 4) and (6, 0), then find the vertices of triangle, centroid and circumcentre of triangle.

Sol. Let points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be vertices of ΔABC.



$x_1 + x_3 = 0,\ y_1 + y_3 = 8$

$x_2 + x_3 = 12,\ y_2 + y_3 = 8$

$x_1 + x_2 = 12,\ y_1 + y_2 = 0$

Solving we get A(0, 0), B (12, 0) and C(0, 8).

Hence ΔABC is right angled triangle $\angle A = \pi/2$.

$\therefore$ Circumcentre is midpoint of hypotenuse which is (6, 4) itself and

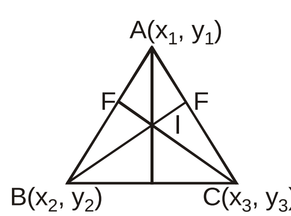
centroid $\equiv \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right) = \left(4, \frac{8}{3}\right)$.

Ex.2 Prove that the incentre of the triangle whose vertices are given by $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ is $\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$ where a, b, and c are the sides opposite to the angles A, B and C respectively.

Sol. By geometry, we know that $\frac{BD}{DC} = \frac{AB}{AC}$ (since AD bisects $\angle A$).

If the length of the sides AB, BC and AC are c, a and b respectively, then $\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}$.

$\Rightarrow$ Coordinates of D are $\left(\frac{bx_2 + cx_3}{b+c}, \frac{by_2 + cy_3}{b+c}\right)$



Since $\frac{BD}{DC} = \frac{c}{b}$, $BD = \frac{ac}{b+c}$

B bisects $\angle B$, Hence $\frac{ID}{IA} = \frac{BD}{BA} = \frac{\left(\frac{ac}{a+c}\right)}{c} = \frac{a}{c+b}$

Let the coordinates of I be $(\overline{x}, \overline{y})$.

Then $\overline{x} = \frac{ax_1 + bx_2 + cx_3}{a+b+c}, \overline{y} = \frac{ay_1 + by_2 + cy_3}{a+b+c}$ (using section formula).

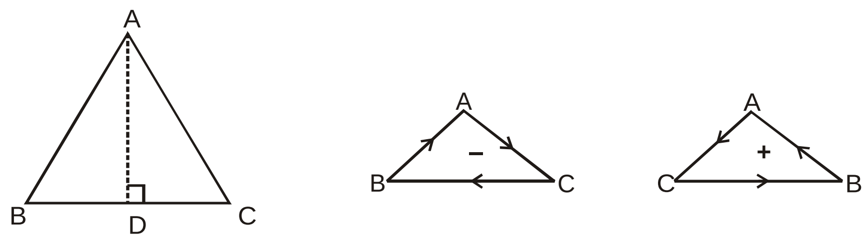
B. AREA OF A TRIANGLE

Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively be the coordinates of the vertices A, B, C of a triangle ABC. Then the area of triangle ABC, is

$$\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] \qquad \text{...(1)}$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \qquad \text{...(2)}$$

While using formula (1) or (2). order of the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) has not been taken into account. If we plot the points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then the area of the triangle as obtained by using formula (1) or (2) will be positive or negative as the points A, B, C are in anti-clockwise or clockwise directions.



So, while finding the area of triangle ABC, we use the formula :

$$\text{Area of } \Delta ABC = \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right| = \text{Modulus of } \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Notes :

(i) If three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear, then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(ii) Equation of straight line passing through (x_1, y_1) and (x_2, y_2) is given by $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

(iii) In case of polygon with (x_1, y_1), (x_2, y_2) .... (x_n, y_n) the area is given by

$$\frac{1}{2}\left|(x_1y_2 - y_1x_2) + (x_2y_3 - y_2x_3) + \text{........} + (x_{n-1}y_n - y_{n-1}x_n) + (x_ny_1 - y_nx_1)\right|$$

Ex.3 The vertices of quadrilateral in order are (–1, 4), (5, 6), (2, 9) and (x, x^2). The area of the quadrilateral is $\boldsymbol{\frac{15}{2}}$ ***sq.*** units, then find the point (x, x^2)

Sol. Area of quadrilateral $= \frac{1}{2}\left|\begin{vmatrix} -1 & 4 \\ 5 & 6 \end{vmatrix} + \begin{vmatrix} 5 & 6 \\ 2 & 9 \end{vmatrix} + \begin{vmatrix} 2 & 9 \\ x & x^2 \end{vmatrix} + \begin{vmatrix} x & x^2 \\ -1 & 4 \end{vmatrix}\right| = \frac{1}{2}\left|-26 + 33 + 2x^2 - 9x + 4x + x^2\right| = \frac{1}{2}\left|3x^2 - 5x + 7\right| = \frac{15}{2}$

$\therefore \quad 3x^2 - 5x + 7 = \pm 15 \qquad \therefore\ 3x^2 - 5x - 8 = 0,\ 3x^2 - 5x + 22 = 0 \Rightarrow x = \frac{8}{3}, x = -1$

Hence point is $\left(\frac{8}{3}, \frac{64}{9}\right)$ or (–1, 1). But (–1, 1) will not form a quadrilateral as per given order of the points. Hence the required point is $\left(\frac{8}{3}, \frac{64}{9}\right)$

(1) Locus : When a point moves in a plane under certain geometrical conditions, the point traces out a path. This path of a moving point is called its locus.

Note : All those points which satisfy the given geometrical condition will definitely lie on the locus. But converse is not true always.

(2) Equation of Locus : The equation to a locus is the relation which exists between the coordinates of any point on the path, and which holds for no other point except those lying on the path.

(3) Procedure for finding the equation of the locus of a point

(i) If we are finding the equation of the locus of a point P, assign coordinates (h, k) to P.

(ii) Express the given conditions as equations in terms of the known quantities to facilitate calculations. We sometimes include some unknown quantities known as parameters.

(iii) Eliminate the parameters. so that the eliminate contains only h, k and known quantities.

(iv) Replace h by x, and k by y, in the eliminate. The resulting would be the equation of the locus of P.

(v) If x and y coordinates of the moving point are obtained in terms of a third variable t (called the parameter), eliminate t to obtain the relation in x and y and simplify this relation. This will give the required equation of locus.

Ex.4 Find the focus of the middle points of the segment of a line passing through the point of intersection of the lines ax + by + c = 0 and lx + my + n = 0 and intercepted between the axes

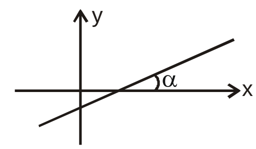
Sol. Any line (say L = 0) passing through the point of intersection of ax + by + c = 0 and lx + my + n = 0 is (ax + by + c) + λ (lx + my + n) = 0, where λ is any real number.

Point of intersection of L = 0 with axes are $\left(-\frac{c+\lambda n}{a+\lambda l}, 0\right)$ and $\left(0, -\frac{c+\lambda n}{b+\lambda m}\right)$

Let the mid point be (h, k). Then $h = -\frac{1}{2}\left(\frac{c+\lambda n}{a+\lambda l}\right)$ and $k = -\frac{1}{2}\left(\frac{c+\lambda n}{b+\lambda m}\right)$

Eliminating λ, we get $\frac{2ah+c}{2hl+c} = \frac{2kb+c}{2km+c}$. The required locus is : 2(am – lb) = (lc – an) x + (nb – mc)y.

(4) Inclination of a line : Its a measure of the smallest non-negative angle which the line makes with +ve direction of the x-axis [angle being measured in anti-clockwise direction]. $0 \le \alpha < \pi$



(5) Slope of the line : If the inclination of line is θ and $\theta \ne \frac{\pi}{2}$ then its slope is defined as $\tan\theta$ and denoted by 'm'

(i) If $\theta = 0$, then m = 0 i.e. line parallel to x-axis.

(ii) If $\theta = 90^o$, then m does not exist i.e. line parallel to y-axis

(iii) Slope of line joining two points $A(x_1, y_1)$ & $B(x_2, y_2)$ is $\boxed{m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}}$

(iv) If a line equally inclined with co-ordinate axes then slope is ± 1.

(6) Intercepts : The point where a line cuts the x-axis (or y-axis) is called its x-intercept (or y-intercept).

(i) Intercepts may be +ve, –ve or zero.

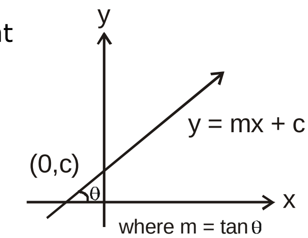
(ii) A line making an intercept of –a with y-axis means the line passing through (0,–a)

(iii) A line makes equal non-zero intercept with both co-ordinate axes then slope is –1.

(iv) A line makes non-zero intercept with both co-ordinate axes equal in magnitude then slope is ±1.

C. STANDARD EQUATIONS OF STRAIGHT LINES

(1) General Form : Any first degree equation of the form Ax + By + C = 0, where A, B, C are constant always represents general equation of a straight line (at least one out of A and B is non zero.)



(2) Slope - Intercept Form :

y = mx + c

where m = slope of the line = $\tan\theta$

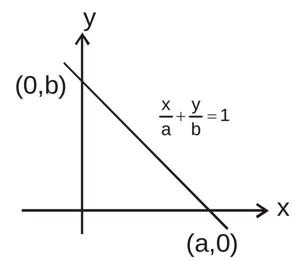
c = y intercept

(3) Intercept Form :

$$\frac{x}{a} + \frac{y}{b} = 1$$

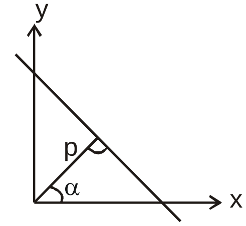
x intercept = a

y intercept = b



(4) Normal Form :

$x\cos\alpha + y\sin\alpha = p$, where α is the angle which the perpendicular to the line makes with the axis of x and ρ is the length of the perpendicular from the origin to the line. p is always positive



(5) Slope Point Form :

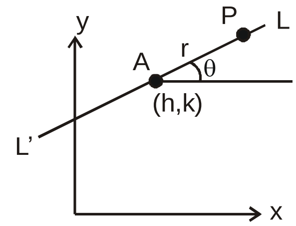
Equation : $y - y_1 = m(x - x_1)$, is the equation of line passing through the point (x_1, y_1) and having the slope 'm'

(6) Two points Form :

Equation : $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, is the equation of line passing through two points (x_1, y_1) and (x_2, y_2)

(7) Parametric Form :

To find the equation of a straight line which passes through a given point A(h, k) and makes a given angle θ with the positive direction of the x-axis. P(x, y) is any point on the line LAL'. Let AP = r, $x - h = r\cos\theta$, $y - k = r\sin\theta$



$\frac{x-h}{\cos\theta} = \frac{y-k}{\sin\theta} = r$ is the equation of the straight line LAL'.

Any point on the line will be of the form $(h + r\cos\theta, k + r\sin\theta)$, where |r| gives the distance of the point P from the fixed point (h, k)

Note : If point P is taken relatively upward to A then r is positive otherwise negative. If line is parallel to x-axis then for he point right to A, r is positive and for left to A, r is negative.

D. REDUCTION OF GENERAL EQUATION TO DIFFERENT STANDARD FORMS

(1) Slope - Intercept Form :

To reduce the equation Ax + By + C = 0 to the form y = mx + c

Given equation is Ax+ By + C = 0 $\Rightarrow$ y = $\frac{-A}{B}$, c = $-\frac{C}{B}$ $(B \neq 0)$

Note : Slope of the line Ax + By + c = 0 is $-\frac{A}{B}$.i.e $-\left(\frac{\text{coefficient of } x}{\text{coefficient of } y}\right)$. y intercept the line $= -\frac{C}{B}$

(2) Intercept Form :

To reduce the equation Ax + By + C = 0 the form $\frac{x}{a} + \frac{y}{b} = 1$. This reduction is possible only when $C \neq 0$

Given equation is Ax + By = – C

$\Rightarrow \frac{x}{\frac{-C}{A}} + \frac{y}{\frac{-C}{B}} = 1$ which is the form $\frac{x}{a} + \frac{y}{b} = 1$. where a = $\frac{-C}{A}$, b = $-\frac{C}{B}$

Note : Intercept on the x-axis = $-\frac{C}{A}$, intercept on the y-axis = $-\frac{C}{B}$. Thus intercept of a straight line on the x-axis can be obtained by putting y = 0 in the equation of the line and then finding the value of x. similarly, intercept on the y-axis can be obtained by putting x = 0 and solving for y.

(3) Normal form :

To reduce the equation Ax + By + C = 0 to the form x cos α + y sin α = p

Given equation is Ax + By + C = 0 or, Ax + By = – C

Case I : When – C > 0, then normal form is $\frac{A}{\sqrt{A^2+B^2}}x + \frac{B}{\sqrt{A^2+B^2}}y = \frac{-C}{\sqrt{A^2+B^2}}$

where cosα = $\frac{A}{\sqrt{A^2+B^2}}$, sin α = $\frac{B}{\sqrt{A^2+B^2}}$; p = $\frac{-C}{\sqrt{A^2+B^2}}$

Case II : When –C < 0, the write the equation as –Ax – By = C

$$\frac{-A}{\sqrt{A^2+B^2}}x + \frac{-B}{\sqrt{A^2+B^2}}y = \frac{C}{\sqrt{A^2+B^2}}$$

where cos α = $\frac{-A}{\sqrt{A^2+B^2}}, \sin\alpha = \frac{-B}{\sqrt{A^2+B^2}}, p = \frac{C}{\sqrt{A^2+B^2}}$

Note : In the normal form x cos α + y sin α = p, p is always taken as positive.

Ex.5 Reduce the line 2x – 3y + 5 = 0, in slope intercept, intercept and normal forms.

Sol. Slope - Intercept Form : y = $\frac{2x}{3} + \frac{5}{3}$, tan$\theta$ = m = $\frac{2}{3}$, c = $\frac{5}{3}$

Intercept Form : $\frac{x}{\left(-\frac{5}{2}\right)} + \frac{y}{\left(\frac{5}{3}\right)} = 1, \ a = -\frac{5}{2}, b = \frac{5}{3}$. Normal Form **:** $-\frac{2x}{\sqrt{13}} + \frac{3y}{\sqrt{13}} = \frac{5}{\sqrt{13}}$

$$\sin\alpha = \frac{3}{\sqrt{13}}, \cos\alpha = \frac{-2}{\sqrt{13}}, p = \frac{5}{\sqrt{13}}$$

Ex.6 Find the equations of the lines which pass through the point (3, 4) and the sum of their respective intercepts on the axes is 14.

Sol. Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$ ...(i)

This passes through (3, 4) therefore $\frac{3}{a} + \frac{4}{b} = 1$ ...(ii)

It is given that a + b = 14 $\Rightarrow$ b = 14 – a

Putting b = 14 – a in (ii), we get $\frac{3}{a} + \frac{4}{14-a} = 1$ $\Rightarrow$ a^2 – 13a + 42 = 0

$\Rightarrow$ (a – 7) (a – 6) = 0 $\Rightarrow$ a = 7, 6 $\Rightarrow$ two such lines are there.

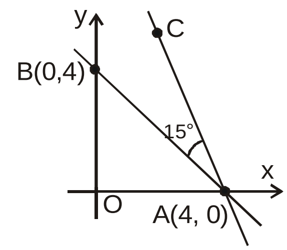
For a = 7, b = 14 – 7 = 7 and for a = 6, a = 14 – 6 = 8

Putting the values of a and b in (i), we get the equations of lines

$\frac{x}{7} + \frac{y}{7} = 1$ and $\frac{x}{6} + \frac{y}{8} = 1$ or x + y = 7 and 4x + 3y = 24.

Ex.7 A rod of steel is fixed at A (4, 0) and a toy is placed on it at B (0, 4). Now rod is rotated about A through an angle of 15° in clockwise direction, then find the new position of a toy.

Sol. Let new position of a toy be C.



Slope of AB = $\frac{4-0}{0-4} = -1 \quad \Rightarrow \theta = 135°$

Rod is rotated through 15° in clockwise

$\theta_{new} = 135° - 15° = 120°$

AB = $4\sqrt{2} \Rightarrow h = 4, k = 0$

Hence C = $(h + r\cos\theta, k = r\sin\theta) = (4 + 4\sqrt{2}\cos 120°, 0 + 4\sqrt{2}\sin 120°) = (4-2\sqrt{2}, 2\sqrt{6})$

Ex.8 If the straight line through the point P(3, 4) makes an angle $\pi/6$ with the x-axis and meets the line $12x + 5y + 10 = 0$ at Q, find the length of PQ.

Sol. The equation of a line passing through P(3, 4) and making an angle = $\pi/6$ with the x-axis is

$\frac{x-3}{\cos\frac{\pi}{6}} = \frac{y-4}{\sin\frac{\pi}{6}}$ = r, where r represents the distance of any point on this line from the given point P(3, 4). The co-ordinates of any point Q on this line are $\left(r\cos\frac{\pi}{6}+3, r\sin\frac{\pi}{6}+4\right)$

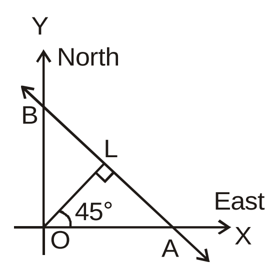
If Q lies on $12x + 15y + 10 = 0$, then $12\left(13 + r\frac{\sqrt{3}}{2}\right) + 5\left(4 + \frac{r}{2}\right) + 10 = 0 \Rightarrow r = \frac{-132}{12\sqrt{3}+5} \Rightarrow$ length PQ= $\frac{132}{12\sqrt{3}+5}$

Ex.9 A canal is $4\frac{1}{2}$ kms from a place and the shortest route from this place to the cenal is exactly north-east. A village is 3 kms north and 4 kms east from the place. Does it lie on canal?

Sol. Let AB be the canal and O be the given place.

Let L be the foot of perpendicular from O to AB.

Given, OL = 9/2. And $\angle AOL = 45°$



$x\cos 45° + y\sin 45° = \frac{9}{2}$

or $x + y = \frac{9}{\sqrt{2}}$ ...(1)

Let S be the given village, then S = (4, 3). Putting x = 4 and y = 3 in equation (1), we get $4 + 3 = \frac{9}{\sqrt{2}}$, which is not true. Thus the co-ordinates of S doesn't satisfy equation (1) and hence the given village does not lie on the canal.

(4) Position of a point w.r.t. to a line L : Ax + By + c = 0

(i) If the points $P(x_1, y_1)$ & $Q(x_2, y_2)$ lies on the same side of the line $Ax + By + C = 0$ then the expressions $Ax_1 + By_1 + C$ & $Ax_2 + By_2 + C$ have same sign otherwise if P and Q lies on opposite side then $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ will have opposite sign.

(ii) If only one point is given then position of that point is checked w.r. to origin.

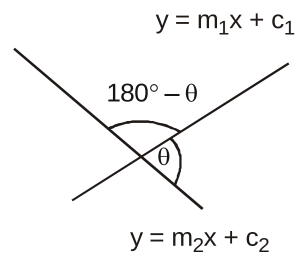
Ex.10 Find the range of θ in the interval $(0, \pi)$ such that the points (3, 5) and $(\sin\theta, \cos\theta)$ lie on the same side of the line $x + y - 1 = 0$

Sol. Here $3 + 5 - 1 = 7 > 0$. Hence $\sin\theta + \cos\theta - 1 > 0 \Rightarrow \sin(\frac{\pi}{4}+\theta) > \frac{1}{\sqrt{2}} \Rightarrow \frac{\pi}{4} < \frac{\pi}{4}+\theta < \frac{3\pi}{4} \Rightarrow 0 < \theta < \frac{\pi}{2}$

E. ANGLE BETWEEN TWO STRAIGHT LINES

If θ is the acute angle between two lines, then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

where m_1 and m_2 are the slopes of the two lines and are finite.



Notes :

(**i**) If the two lines are perpendicular to each other then $m_1 m_2 = -1$

(ii) Any line perpendicular to $ax + by + c = 0$ is of the form $bx - ay + k = 0$

(iii) If the two lines are parallel or coincident, then $m_1 = m_2$

(iv) Any line parallel to $ax + by + c = 0$ is of the form $ax + by + k = 0$

(v) If any of the two lines is perpendicular to x-axis, then the slope of that line is infinite.

Let $m_1 = \infty$, Then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{1 - \frac{m_2}{m_1}}{\frac{1}{m_1} + m_2}\right| = \left|\frac{1}{m_2}\right|$ or $\theta = |90° - \alpha|$, where $\tan\alpha = m_2$

i.e. angle θ is the complimentary to the angle which the oblique line makes with the x-axis.

(vi) If lines are equally inclined to the coordinate axis then $m_1 + m_2 = 0$

Ex.11 Find the equation to the straight line which is perpendicular bisector of the line segment AB, where A, B are (a,b) and (a', b') respectively.

Sol. Equation of AB is $y - b = \frac{b'-b}{a'-a}(x - a)$ i.e. $y(a' - a) - x(b' - b) = a'b - ab'$.

Equation to the line perpendicular to AB is of the form $(b' - b)y + (a' - a)x + k = 0$ ....(1)

Since the midpoint of AB lies on (1), $(b'-b)\left(\frac{b+b'}{2}\right) + (a'-a)\left(\frac{a+a'}{2}\right) + k = 0$

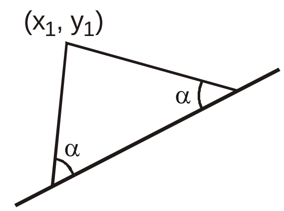
Hence the required equation of the straight line is $2(b'-b)y + 2(a'-a)x = (b'^2 - b^2 + a'^2 - a^2)$

(1) Equation of straight Lines passing through a given point and equally inclined to a given line :

Let the straight passing through the point (x_1, y_1) and make equal angles with the given straight line $y = mx + c$. If m is the slope of the required line and α is the angle which this line makes with the given line then $\tan\alpha = \pm\frac{m_1 - m}{1 + m_1 m}$

(2) The above expression for $\tan\alpha$, given two values of m, say m_A and m_B. The required equations of the lines through the point (x_1, y_1) and making equal angles α with the given line are

$y - y_1 = m_A(x - x_1)$, $y - y_1 = m_B(x - x_1)$

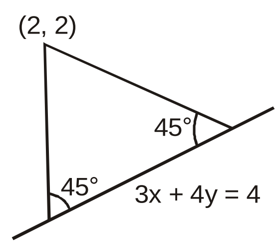


Ex.12 Find the equation to the sides of an isosceles right-angled triangled, the equation of whose hypotenuse is $3x + 4y = 4$ and the opposite vertex is the point (2, 2).

Sol. The problem can be restarted as :

Find the equation to the straight lines passing through the given point (2, 2) and making equal angles of 45° with the given straight line $3x + 4y - 4 = 0$. Slope of the line $3x + 4y - 4 = 0$ is $m_1 = -\frac{3}{4}$

$\Rightarrow \tan 45° = \pm\frac{m - m_1}{1 + m_1 m}$, i.e., $1 = \pm\frac{m + 3/4}{1 - \frac{3}{4}m}$



$m_A = \frac{1}{7}$, and $m_B = -7$

Hence the required equations of the two lines are

$y - 2 = m_A(x - 2)$ and $y - 2 = m_B(x - 2)$ $\Rightarrow 7y - x - 12 = 0$ and $7x + y = 16$.

F. DISTANCE BETWEEN POINT & LINE AND TWO PARALLEL LINES

(1) Length of the Perpendicular from a Point on a Line :

The length of the perpendicular from $P(x_1, y_1)$ on $ax + by + c = 0$ is $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

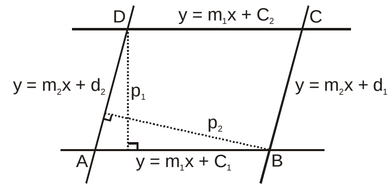
The length of the perpendicular from origin on $ax + by + c = 0$ is $\left|\frac{c}{\sqrt{a^2 + b^2}}\right|$

(2) The distance between two parallel lines :

The distance between two parallel lines : $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $\frac{|c_1 - c_2|}{\sqrt{a^2 - b^2}}$

(3) Area of parallelogram with given sides :

$$\text{Area} = \left|\frac{(C_1 - C_2)(d_1 - d_2)}{m_1 - m_2}\right|$$

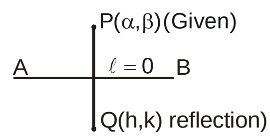


(4) Condition of parallelogram as shown becomes a rhombus :

$$p_1 = p_2 \Rightarrow \left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right| = \left|\frac{d_1 - d_2}{\sqrt{a^2 + b^2}}\right|$$

(5) Reflection (Image) of a point P(α, β) about a line (ax + by + c = 0)

$$\frac{x - \alpha}{a} = \frac{y - \beta}{b} = -\frac{2(a\alpha + b\beta + c)}{a^2 + b^2}$$



(6) Foot of perpendicular from a point (α, β) to a given line $\ell \equiv$ ax + by + c = 0

$$\frac{x - \alpha}{a} = \frac{y - \beta}{b} = -\frac{(a\alpha + b\beta + c)}{a^2 + b^2}$$

(7) Shifting of the origin :

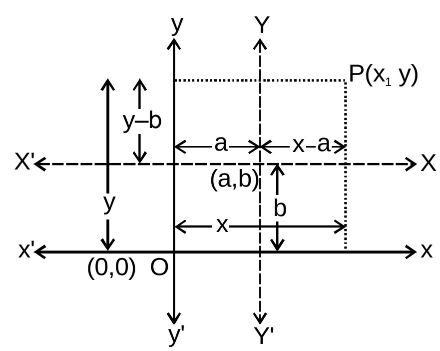
x, y $\Rightarrow$ old co-ordinates axes

X, Y $\Rightarrow$ New co-ordinate axes

X = 0 $\Rightarrow$ x – a = 0 $\Rightarrow$ x = a

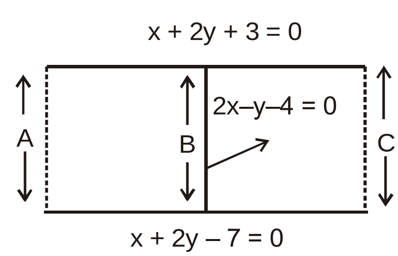
Y = 0 $\Rightarrow$ y – b = 0 $\Rightarrow$ y = b

Slope and area of closed figure remains unchanged under the translation of co-ordinate axes.



Ex.13 Three lines x + 2y + 3 = 0, x + 2y – 7 = 0 and 2x – y – 4 = 0 form 3 sides of two squares. Find the equation of remaining sides of these squares.

Sol. Distance between the two parallel lines is $\frac{|7+3|}{\sqrt{5}} = 2\sqrt{5}$.



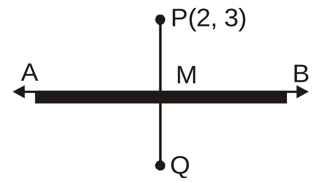
The equations of sides A and C are of the form 2x – y + k = 0. Since distance between sides A and B = distance between sides

B and C $\frac{|k-(-4)|}{\sqrt{5}} = 2\sqrt{5}$ $\Rightarrow$ $\frac{k+4}{\sqrt{5}} = \pm 2\sqrt{5}$ $\Rightarrow$ k = 6, –14

Hence the fourth sides of the two squares are (i) 2x – y + 6 = 0, (ii) 2x – y – 14 = 0

Ex.14 Find the foot of the perpendicular drawn from the point (2, 3) to the line 3x − 4y + 5 = 0. Also, find the image of (2, 3) in the given line.

Sol. Let $AB \equiv 3x - 4y + 5 = 0$, $P \equiv (2,3)$ and $PM \perp AB$.



Slope of $AB = \frac{3}{4}$ $\Rightarrow$ slooe of $PM = \frac{-4}{3} = \tan\theta$ (say)

$\Rightarrow$ $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{-3}{3} = \tan\theta$ (say) $\Rightarrow$ $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{-3}{5}$

Now, $r = p = \frac{3\times 2 - 4\times 3 + 5}{\sqrt{9+16}} = \frac{6-12+5}{5} = \frac{-1}{5}$

Which is the foot of the perpendicular. $\Rightarrow$ $M = \left(2 - \frac{1}{5}\cos\theta,\ \ 3 - \frac{1}{5}\sin\theta\right) = \left(\frac{53}{25}, \frac{71}{25}\right)$

Let Q be the image of P $\Rightarrow$ $Q = \left(2 - \frac{2}{5}\cos\theta,\ 3 - \frac{2}{5}\sin\theta\right) = \left(\frac{56}{25}, \frac{67}{25}\right)$

G. BISECTORS OF THE ANGLES BETWEEN TWO GIVEN LINES

Angular bisector is the locus of a point which moves in such a way so that its distance from two intersecting lines remains same.

The equation of the two bisectors of the angles between the lines $a_1x + b_1y + c_1 = 0$ and

$a_2 x + b_2y + c_2 = 0$ are $\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$.

If the two given lines are not perpendicular i.e. $a_1 a_2 + b_1 b_2 \neq 0$, then one of these equation is the equation of the bisector of the acute angle and the other that of the obtuse angle.

Note : Whether both lines are perpendicular or not but the angular bisectors of these lines will always be mutually perpendicular.

(1) The bisectors of the acute and the obtuse angles :

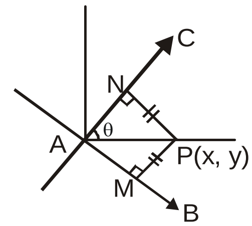
Take one of the lines angle let its slope be m_1 and take one of the bisectors and let its slope be m_2.

If θ be the acute angle between them, then find $\text{tna}\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$.

If $\tan\theta > 1$ then the bisector taken is the bisector of the obtuse angle and the other one will be the bisector of the acute angle. If $0 < \tan\theta < 1$ then the bisector taken is the bisector of the acute angle and the other one will be the bisector of the obtuse angle.

If two lines are $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, then

$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$ will represent the equation of the bisector of the acute or obtuse angle between the lines according as $c_1c_2(a_1a_2+b_1b_2)$ is negative or positive.



(2) The equation of the bisector of the angle containing the origin

Write the equations of the two lines so that the constants c_1 and c_2 become positive. Then the equation $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$ is the equation of the bisector containing the origin.

Notes :

(i) If $a_1a_2+b_1b_2<0$, then the origin will lie in the acute angle and if $a_1a_2+b_1b_2>0$, then origin will lie in the obtuse angle.

(ii) The note (i) is helpful in finding the equation of bisector of the obtuse angle or acute angle directly.

(3) The equation of the bisector of the angle which contains a given point

The equation of the bisector of the angle between the two lines containing the point (α,β) is

$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=+\left(\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}\right)$ or $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=-\left(\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}\right)$ according as $a_1\alpha+b_1\beta+c_1$ and $a_2\alpha+b_2\beta+c_2$ are of the same sings or of opposite signs.

Ex.15 For the straight line $4x+3y-6=0$ and $5x+12y+9=0$, find the equation of the

(i) bisector of the obtuse angle between them.

(ii) bisector of the actus angle between them.

(iii) bisector of the angle which contains (1, 2).

Sol. Equations of bisectors of the angles between the given lines are

$\frac{4x+3y-6}{\sqrt{4^2+3^2}}=\pm\frac{5x+12y+9}{\sqrt{5^2+12^2}}\Rightarrow 9x-7y-41=0$ and $7x+9y-3=0$.

If θ is the acute angle between the line $4x+3y-6=0$ and the bisector $9x-7y-41=0$, then

$$\tan\theta=\left|\frac{-\frac{4}{3}-\frac{9}{7}}{1+\left(\frac{-4}{3}\right)\frac{9}{7}}\right|=\frac{11}{3}>1$$

Hence

(i) The bisector of the obtuse angle is $9x-7y-41=0$

(ii) The bisector of the acute angle is $7x+9y-3=0$

(iii) The bisector of the angle containing the origin $\frac{-4x-3y+6}{\sqrt{(-4)^2+(-3)^2}} = \frac{5x+12y+9}{\sqrt{5^2+12^2}} \Rightarrow 7x+9y-3=0$

(i) For the point (1, 2), 4x + 3y − 6 = 4 × 1 + 3 × 2 − 6 > 0 ⇒ 5x + 12y + 9 = 5 × 1 + 12 × 2 + 9 > 0
Hence equation of the bisector of the angle containing the point (1, 2) is

$$\frac{4x+3y-6}{5} = \frac{5x+12y+9}{13} \Rightarrow 9x-7y-41=0$$

Alternative : 5 lines. Similarly bisector of obtuse angle is 9x − 7y − 41 = 0.

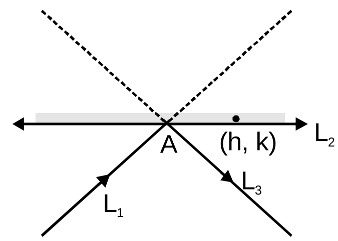
(4) The equation of reflected ray :

Let $L_1 \equiv a_1x+b_1y+c_1=0$ be the incident ray in the line mirror $L_2 \equiv a_2x+b_2y+c_2=0$

Let L_3 be the reflected ray from the line L_2. Clearly L_2 will be one of the bisectors of the angles between L_1 and L_3. Since L_3 passes through A, so $L_3 \equiv L_1 + \lambda L_2 = 0$.

Let (h, k) be a point on L_2. Then,

$$\frac{|a_1h+b_1k+c_1|}{\sqrt{a_1^2+b_1^2}} = \frac{|a_1h+b_1k+c_1+\lambda(a_2h+b_2k+c_2)|}{\sqrt{(a_1+\lambda a_2)^2+(b_1+\lambda b_2)^2}}$$



Since (h, k) lies on L_2, $a_2h + b_2k + c_2 = 0$

$\Rightarrow \quad a_1^2 + a_2^2\lambda^2 + 2a_1a_2\lambda + b_1^2 + b_2^2\lambda^2 + 2b_1b_2\lambda + a_1^2 + b_1^2$

$\Rightarrow \quad \lambda = 0$ or $\lambda = \frac{-2(a_1a_2+b_1b_2)}{a_2^2+b_2^2}$

But λ = 0 gives $L_3 = L_1$. Hence $L_3 = L_1 - \frac{2(a_1a_2+b_1b_2)}{a_2^2+b_2^2}\; L_2 = 0$.

Note : Some times the reflected ray L_3 is also called the mirror image of L_1 in L_2.

H. FAMILY OF LINES

The general equation of the family of lines through the point of intersection of two given lines is L + λL' = 0, where L = 0 and L' = 0 are the two given lines, and λ is a parameter. Conversely, any line of the form L_1 + λ, L_2 = 0 passes through a fixed point which is the point of intersection of the lines L_1 = 0 and L_2 = 0.

The family of lines perpendicular to a given line ax + by + c = 0 is given by bx − ay + k = 0, where k is a parameter.

The family of lines parallel to a given line ax + by + c = 0 is given by ax + by + k = 0, where k is a parameter.

Ex.16 Show that all the chords of the curve $3x^2 - y^2 - 2x + 4y = 0$, which subtend a right angle at the origin pass through a fixed point. Find that the point.

Sol. Let the equation of chord be lx + my = 1. So equation of pair of straight line joining origin to the points of intersection of chord and curve.

$3x^2 - y^2 - 2x(lx + my) + 4y(l + my) = 0$, which subtends right angle at origin.

⇒ (3 − 2l + 4m − 1) = 0 ⇒ 1 = 2m + 1. Hence chord becomes (2m + 1)x + my = 1

x − 1 + m(2x + y) = 0

L_1 $\qquad$ L_2 $\qquad$ Which will pass through point of intersection of L_1 = 0 and L_2 = 0.

⇒ x = 1, y = − 2. Hence fixed point is (1, − 2).

(1) One Parameter Family of Straight Lines

If a linear expression L_1 contains an unknown coefficient, then the line $L_1 = 0$ can not be a fixed line. Rather it represents a family of straight lines known as one parameter family of straight lines. e.g. family of lines parallel to the x-axis i.e. y = c and family of straight lines passing through the origin i.e. y = mx.

Each member of the family passes a fixed point. We have two methods to find the fixed point.

Method (i) :

Let the family of straight lines of the form ax + by + c = 0 where a, b, c are variable parameters satisfying the condition al + bm + cn = 0, where l, m, n, are given and $n \neq 0$. Rewriting the condition as $a\left(\frac{1}{n}\right)+b\left(\frac{m}{n}\right) + c = 0$ and comparing with the given family of straight lines, we find that each member of it passes through the fixed point $\left(\frac{1}{n},\frac{m}{n}\right)$

Ex.17 If the algebraic sum of perpendiculars from n given points on a variable straight line is zero then prove that the variable straight line passes through a fixed point.

Sol. Let n given points be (x_1, y_1) where i = 1, 2 .... n and the variable line is ax + by + c = 0, Given that $\sum_{i=1}^{n}\left(\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right) = 0$. $\Rightarrow a\Sigma x_1 + a\Sigma y_1 + cn = 0 \Rightarrow a\frac{\sum x_i}{n}+b\frac{\sum y_i}{n} + c = 0.$

Hence the variable straight line always passes through the fixed point $\left(\frac{\sum x_i}{n}.\frac{\sum y_i}{n}\right)$.

Method (ii) :

If a family of straight lines can be written as $L_1 + \lambda L_2 = 0$ where L_1, L_2 are two fixed lines and λ is a parameter, then each member of it will pass through a fixed point given by point of intersection of $L_1 = 0$ and $L_2 = 0$.

Note : If $L_1 = 0$ an $L_2 = 0$ are parallel lines, they will meet at infinity.

Ex.18 Prove that each member of the family of straight lines

$(3 \sin\theta + 4 \cos\theta)x + (2 \sin\theta - 7\cos\theta) + (\sin\theta + 2\cos\theta) = 0$ (θ is a parameter)

passes through a fixed point.

Sol. The given family of straight lines can be rewritten as $(3x + 2y + 1) \sin\theta + (4x - 7y + 2) \cos\theta = 0$

or, $(4x - 7y + 2) + \tan\theta\,(3x + 2y + 1) = 0$ which is of the form $L_1 + \lambda L_2 = 0$

Hence each member of it will pass through a fixed point which is the intersection of 4x − 7y + 2 = 0 and 3x + 2y + 1 = 0 i.e. $\left(\frac{-11}{29},\frac{2}{29}\right)$

(2) Concurrency of Straight Lines : The condition for 3 lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2 + c_2 = 0$, $a_3x + b_3x + b_3y + c_3 = 0$ to be concurrent is

(i) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

(ii) There exist 3 constants l, m, n (not all zero at the same time) such that $L_1 + mL_2 + nL_3 = 0$, where $L_1 = 0$, $L_2 = 0$ and $L_3 = 0$ ar the three given straight lines.

(iii) the three lines are concurrent if any one of the lines passes through the pint of intersection of the other two lines.

I. PAIR OF STRAIGHT LINES

The combined equation of pair of straight lines $L_1 = a_1x + b_1y + c_1 = 0$ and $L_2 = a_2x + b_2y + c_2 = 0$ is $(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0$ i.e. $L_1L_2 = 0$. Opening the brackets and comparing the terms with the terms of general equation of 2nd degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we can get all the following results for a pair of straight lines.

The general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$ and $h^2 \geq ab$. $\Rightarrow$ $abc + 2fgh - al^2 - bg^2 - ch^2 = 0$ and $h^2 \geq ab$.

The homogeneous second degree equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines through the origin if $h^2 \geq ab$.

If the lines through the origin whose joint equation is $ax^2 + 2hxy + by^2 = 0$, are $y = m_1x$ and $y = m_2x$, then $y^2 - (m_1 + m_2)xy + m_1m_2x^2 = 0$ and $y^2 + \frac{2h}{b}xy + \frac{a}{b}x^2 = 0$ are identical, so that

$m_1 + m_2 = -\frac{2h}{b}$, $m_1m_2 = \frac{a}{b}$

If θ be the angle between two lines, through the origin, then $\tan\theta = \pm \frac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1+m_1m_2} = \pm \frac{2\sqrt{h^2 - ab}}{a+b}$

The lines are perpendicular if $a + b = 0$ and coincident if $h^2 = ab$.

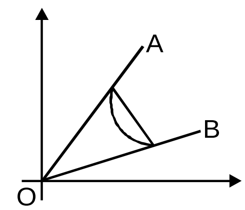
In the more general case, the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ will be perpendicular if $a + b = 0$, parallel if the terms of second degree make a perfect square i.e. $ax^2 + 2hxy + by^2$ gets converted into $(l_1x + 2hxy + by^2$ gets converted into $(l_1x \pm m_1y)^2$, coincident if the whole equation makes a perfect square i.e. $ax^2 + 2hxy + by^2 + 2fy + c$ can be written as $(lx + my + n)^2$.

Note : Point of intersection of the two lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is obtained by solving the equations $\frac{\partial f}{\partial x} = ax + hy + g = 0$ and $\frac{\partial f}{\partial y} = hx + by + f = 0$ where $\frac{\partial f}{\partial x}$ denotes the derivative of f with respect to y, keeping x constant. The fact can be used in splitting $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ into equations of two straight lines. With the above method, the point of intersection can be found. Now only the slopes need to be determined.

If should be noted that the line $ax + hy + g = 0$ and $hx + by + f = 0$ are not the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. These are the lines concurrent with the lines represented by given equation.

(Homogenization) Joint equation of pair of lines joining the origin and the points of intersection of a curve and a line :

If the line $lx + my + n = 0$, $((n \neq 0)$ i.e. the line does not pass through origin) cuts the curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ at two points A and B, then the joint equation of straight lines passing through A and B and the origin is given by homogenizing the equation of the curve by the equation of the line. i.e.



$ax^2 + 2hxy + by^2 + (2gx + 2gy)\left(\frac{\ell x + my}{-n}\right) + c\left(\frac{\ell x + my}{-n}\right)^2 = 0$

is the equation of the line OA and OB.

Ex.19 If the lines $ax + y + 1 = 0$, $x + by + 1 = 0$ and $x + y + c = 0$(a, b and c being distinct and different from 1) are concurrent, then prove that $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{a-c} = 1$.

Sol. Since the given lines are concurrent $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$.

Operating $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get $\begin{vmatrix} a & 1-a & 1-a \\ 1 & b-1 & 0 \\ 1 & 0 & c-1 \end{vmatrix} = 0$

$\Rightarrow$ a (b − 1) (c − 1) − (b − 1) (1 − a) −(c − 1) (1 − 0) = 0 $\Rightarrow \frac{a}{1-a} + \frac{1}{a-b} + \frac{1}{1-c} = 0 \Rightarrow \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$.

Ex.20 The chord $\sqrt{6}\, y = \sqrt{8}\, \rho x + \sqrt{2}$ of the curve $\rho y^2 + 1 = 4x$ subtends a right angle at origin then find the value of ρ.

Sol. $\sqrt{3}y - 2\rho x = 1$ is the given chord. Homogenizing the equation of the curve, we get,

$\rho y^2 - 4x(\sqrt{3}\, y - 2\rho x) + (\sqrt{3}\, y - 2\rho x)^2 = 0 \Rightarrow (4\rho^2 + 8\rho)x^2 + (\rho + 3)y^2 - 4\sqrt{3}\, xy - 4\sqrt{3}\, \rho xy = 0$

Now, angle at origin is 90º $\quad\therefore\quad$ coefficient of x^2 + coefficient of $y^2 = 0$

$\therefore\quad 4\rho^2 + 8\rho + \rho + 3 = 0 \Rightarrow 4\rho^2 + 9\rho + 3 = 0 \quad\therefore\quad \frac{-9 \pm \sqrt{81-48}}{8} = \frac{-9 \pm \sqrt{33}}{8}$.